DIRICHLET SERIES AS A GENERALIZATION OF POWER SERIES

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"All generalizations are dangerous, even this one."

—Alexandre Dumas

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CHAPTER I

INTRODUCTION

The infinite series

$$\sum_{n=0}^{\infty} a_n x^n,$$

in which x is a real variable and the coefficients a_n are real numbers, is called a power series, since the variable x is raised to increasing integral powers. A student first encounters power series in the calculus and soon sees what a basic concept this is in many fields of mathematics. A power series will converge at least for $x = 0$, and some power series will converge in a finite interval symmetric to $x = 0$, and still others will converge for all real values of x. For the values of x in the interval of convergence, whether the interval be degenerate (i.e., [0]), finite, or infinite the power series defines a function f such that:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

In 1715 Brook Taylor [38] showed that the coefficients a_n of the power series that defines the function f can be determined by the formula:

$$a_n = \frac{f^{(n)}(0)}{n!} \ .$$

Therefore, the power series and the values of the function f take on the following form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

and the power series also becomes known as the Taylor series.

Power series were later generalized to the series

$$\sum_{n=0}^{\infty} a_n z^n$$

in which z is a complex variable and the coefficients a_n are complex numbers. This more general series is also called a power series. The region of convergence now becomes the interior of a circle centered at the origin with radius zero, finite, or infinite as the case may be.

By making the transformation $z = w - w_0$ we can transform the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

whose circle of convergence is centered at the origin into the equivalent series

$$\sum_{n=0}^{\infty} a_n (w-w_0)^n$$

which we shall also call a power series in which the circle of convergence is centered at the point $w = w_0$. Therefore, in theorems about power series there is usually no loss in generality in considering the series with circle of convergence centered at the origin.

For the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

with finite, nonzero radius of convergence R, we can make the transformation $z = \frac{w}{R}$ and transform the series into an equivalent power series

$$\sum_{n=0}^{\infty} a_n \left(\frac{1}{R}\right)^n w^n$$

with radius of convergence 1. Therefore, in theorems about power series with finite, nonzero radius of convergence, there is usually no loss in generality in considering the radius of convergence to be 1.

In 1839 Peter G. L. Dirichlet introduced the series

$$\sum_{n=1}^{\infty} a_n n^{-x}$$

in which x is a real variable and the coefficients a_n are complex numbers. This series and generalizations of it became known as Dirichlet series. The present series is called an ordinary Dirichlet series. Series of the form

$$\sum_{n=1}^{\infty} a_n n^{-z}$$

in which z is a complex variable and the coefficients a_n are complex numbers are also called ordinary Dirichlet series. Theorems concerning these series were first recorded by J. Jensen in 1884 [18] and E. Cahen in 1894 [7].

The series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

in which z is a complex variable, the coefficients a_n are complex numbers, and the exponents λ_n are nonnegative, real numbers which satisfy the conditions:

$$\lambda_n < \lambda_{n+1} \ (n=1,2,\ldots),$$

$$\lim_{n\to\infty} \lambda_n = \infty ,$$

is called the general Dirichlet series. If $\lambda_n = \ln n$, then we have

$$e^{-\lambda_n z} = e^{-(\ln n)z} = e^{\ln(n)^{-z}} = n^{-z} ,$$

and, therefore, the ordinary Dirichlet series is a special case of the general Dirichlet series.

If in the general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

we make the transformation $w = e^{-z}$, we will obtain the series

$$\sum_{n=1}^{\infty} a_n w^{\lambda_n}$$

which is a series expansion in increasing, nonnegative powers of w. If $\lambda_n = n - 1$ and we define $a'_{n-1} = a_n$ then this series is a power series,

$$\sum_{n=1}^{\infty} a_n w^{\lambda_n} = \sum_{n=1}^{\infty} a'_{n-1} w^{n-1} = \sum_{n=0}^{\infty} a'_n w^n$$

Therefore, a Dirichlet series can be regarded as a generalization of a power series to unbounded, increasing, nonnegative powers which are otherwise arbitrary.

For the special case of the Dirichlet series in which $\lambda_n = n - 1$ and $a'_{n-1} = a_n$ and thus corresponding to a power series

$$\sum_{n=0}^{\infty} a'_n w^n$$

the transformation $z = -\ln w$ sends the circle of convergence onto a vertical line with abscissa $\sigma = +\infty$, a finite constant, or $-\infty$ depending on whether the radius of convergence of the power series is 0, a finite constant, or $+\infty$, respectively. The interior of the circle of convergence is sent onto the right half-plane bounded on the left by the vertical line with abscissa σ. Just as the power series

$$\sum_{n=0}^{\infty} a'_n w^n$$

converges to a function g, so the corresponding Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges to a function f in the right half-plane bounded on the left by the vertical line with abscissa σ. This vertical line is called the line or axis of convergence, and the abscissa σ is called the abscissa of convergence of the Dirichlet series. Similarly, in the case of the general Dirichlet series, the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges to a function f in a right half-plane bounded on the left by the line of convergence.

For the general Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

with finite abscissa of convergence $\sigma = c$, we can make the transformation

$z = w - c$ and transform the series into an equivalent Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n c} e^{-\lambda_n w}$$

in which the abscissa of convergence is $\sigma = 0$. Therefore, in theorems about Dirichlet series with finite abscissa of convergence, there is usually no loss in generality in considering the abscissa of convergence to be $\sigma = 0$.

The general Dirichlet series was further generalized by J. F. Ritt in 1917 to the case of the series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

in which z is a complex variable, the coefficients a_n are complex numbers, and the exponents λ_n are complex numbers which satisfy the conditions:

$$\lambda_n = \mu_n + \nu_n i,$$

$$\mu_1 \geq 0,$$

$$\mu_n < \mu_{n+1},$$

$$\lim_{n\to\infty} \mu_n = \infty,$$

$$\lim_{n\to\infty} \frac{\nu_n}{\mu_n} = 0.$$

Such series are called Dirichlet series with complex exponents. In this paper we will limit our generalizations to the general Dirichlet series.

The hybrid series

$$\sum_{n=1}^{\infty} a_n z^n e^{-\lambda_n z}$$

in which z is a complex variable, the coefficients a_n are complex numbers, and the exponents λ_n are complex numbers is called, appropriately enough, the Taylor-Dirichlet series [6]. Also, this variation will receive no further attention in this paper.

Since the beginning of this century, Dirichlet series have been subject to intensive study and investigations. These series have provided the motivation for the theories of Riemann-Zeta functions, of modular forms, and of almost periodic functions. They have also motivated theorems of closure, of composition, and of gap and density and have been intimately related to the theories of Laplace-Stieltjes and Mellin-Stieltjes integrals.

Researches done on Dirichlet series proper may be classified in some five distinct groups of the main theory. These may be enumerated as follows:

(1) the classical theory of Dirichlet series, which is concerned above all with obtaining theorems having applications in the analytic theory of numbers;

(2) the theory developed on the basis of the functional equation satisfied by the Riemann-Zeta function defined by the ordinary Dirichlet series

$$\sum_{n=1}^{\infty} n^{-z};$$

(3) that part of the theory which is closely related to certain branches of the theory of functions of a real variable, especially that of the theory of Fourier series, having its origin in the works of Bohr on almost periodic functions;

(4) the theory which is concerned mainly with those properties of the Dirichlet series which provide interest from the point of view of the theory of functions of a complex variable;

(5) the theory of convergence and summability of Dirichlet series [28, p.146].

In view of the fact that Dirichlet series are a generalization of power series, then we would expect many of the theorems concerning power series to generalize to the case of Dirichlet series. This will indeed be the case, however, along with the generalization come some interesting unexpected results.

CHAPTER II

FUNDAMENTAL THEOREMS AND CONCEPTS

We have previously stated that the region of convergence of a power series is the interior of a circle. This fact is established in the following theorem:

<u>Cauchy-Hadamard Theorem [26, p.344]</u>.

Given the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

let $R = \frac{1}{\Lambda}$ where $\Lambda = \limsup_{n\to+\infty} \sqrt[n]{|a_n|}$ and let γ be the circle $|z| = R$, with interior $I(\gamma)$ and exterior $E(\gamma)$. Then there are three possibilities:

1. If $R = 0$, the series is divergent for all $z \neq 0$;
2. If $0 < R < +\infty$, the series is absolutely convergent for all $z \in I(\gamma)$ and divergent for all $z \in E(\gamma)$;
3. If $R = +\infty$, the series is absolutely convergent for all finite z.

For the general Dirichlet series we have the interesting result corresponding to the Cauchy-Hadamard theorem that there is, in general, one half-plane of convergence and a possibly more restrictive half-plane of absolute convergence. We will express this in the following theorem which gives the Dirichlet series generalization of the Cauchy-Hadamard theorem:

A Generalization of the Cauchy-Hadamard Theorem to Dirichlet Series [19].

Given the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

let

$$C = \limsup_{r\to+\infty} \frac{\ln\left|\sum_{[r]\le\lambda_n<r} a_n\right|}{r}$$

where [r] is the greatest integer less than or equal to r and $[r] \le \lambda_n < r$ indicates summation over all the indices n such that $[r] \le \lambda_n < r$ although in some cases this is vacuous and the sum is 0. If C is nonnegative it may also be obtained by the formula

$$C = \limsup_{n\to+\infty} \frac{\ln\left|\sum_{k=1}^{n} a_k\right|}{\lambda_n},$$

and, if C is nonpositive, it may be obtained by the formula

$$C = \limsup_{n\to+\infty} \frac{\ln\left|\sum_{k=n}^{\infty} a_k\right|}{\lambda_n}.$$

Let V denote the vertical line with abscissa C. Let R(V) denote the right half-plane $\mathrm{Re}(z) > C$ and $L(V)$ denote the left half-plane $\mathrm{Re}(z) < C$. Then there are three possibilities:

1. If $C = +\infty$, the series diverges for all finite z.
2. If $-\infty < C < +\infty$, the series converges for all $z \in R(V)$ and diverges for all $z \in L(V)$.
3. If $C = -\infty$, the series converges for all finite z.

Now, let

$$A = \limsup_{n\to+\infty} \frac{\ln \sum\limits_{[r]\le\lambda_n<r} |a_n|}{r} .$$

If A is nonnegative, it may also be obtained by the formula

$$A = \limsup_{n\to+\infty} \frac{\ln \sum\limits_{k=1}^{n} |a_k|}{\lambda_n} ,$$

and, if A is nonpositive, it may be obtained by the formula

$$A = \limsup_{n\to+\infty} \frac{\ln \sum\limits_{k=n}^{\infty} |a_k|}{\lambda_n}$$

Let W denote the vertical line with abscissa A.

Let $R(W)$ denote the right half-plane $\mathrm{Re}(z) > A$ and $L(W)$ denote the left half-plane $\mathrm{Re}(z) < A$.

Then there are three possibilities:

1. If $A = +\infty$, the series fails to converge absolutely for all finite z.
2. If $-\infty < A < +\infty$, the series converges absolutely for all $z \in R(W)$ and fails to converge absolutely for all $z \in L(W)$.
3. If $A = -\infty$, the series converges absolutely for all finite z.

It is well to note that for every Dirichlet series we have

$$0 \leq A - C \leq \limsup_{n \to +\infty} \frac{\ln n}{\lambda_n} . \quad [25, p.168]$$

In the theory of power series we find that the circle of convergence is also the circle of uniform convergence as established by the following theorem:

Uniform Convergence Theorem [26, p.348].

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Then the series is uniformly convergent on every compact subset of the interior of the unit circle.

The theorem above carries over directly to the case of Dirichlet series as is given in the following theorem:

First Generalization of the Uniform Convergence Theorem to Dirichlet Series [25, p.169].

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence $C = 0$. Then the series is uniformly convergent on every compact subset of the right half-plane $\mathrm{Re}(z) > 0$.

However, since circular neighborhoods of the origin in the case of power series correspond to right half-planes in the case of Dirichlet series, the theorem does not carry over directly with this kind of generalization. Instead we may obtain a right half-plane of uniform convergence which is different still from the half-plane of convergence and the half-plane of absolute convergence.

Second Generalization of the Uniform Convergence Theorem to Dirichlet Series [20,5].

Given the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

let T_r denote the least upper bound of

$$\left| \sum_{[r]\le\lambda_n<r} a_n e^{-\lambda_n ti} \right|$$

when the real variable t varies between $-\infty$ and $+\infty$ and r is a continuous, positive, real variable. Now let

$$U = \limsup_{r\to+\infty} \frac{\ln T_r}{r} .$$

If U is nonnegative, then it may also be obtained by the formula,

$$U = \limsup_{n\to+\infty} \frac{\ln T_n}{\lambda_n}$$

where T_n is the least upper bound of

$$\left| \sum_{k=1}^{n} a_k e^{-\lambda_k ti} \right|$$

when the real variable t varies between $-\infty$ and $+\infty$. Let M denote the vertical line with abscissa U. Let R(M) denote the right half-plane $Re(z) > U$ and L(M) denote the left half-plane $Re(z) < U$. Then there are three possibilities:

1. If $U = +\infty$, the series fails to converge uniformly on every right half-plane.
2. If $-\infty < U < +\infty$, the series converges uniformly on every right half-plane strictly contained in R(M) and fails to converge uniformly on every right half-plane P such that $L(M) \cap P \neq \phi$.
3. If $U = -\infty$, the series converges uniformly on every right half-plane.

It is well to note that for every Dirichlet series we have

$$-\infty \leq C \leq U \leq A \leq +\infty \text{ [25, p.169]}$$

where, of course, equality cannot hold in every instance.

Inside the circle of convergence the power series converges to a function f. The question is then raised as to whether a different power series can converge to that same function f inside the circle of convergence. The answer is negative as is established by the following theorem:

Uniqueness Theorem [26, p.350].

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series which converges to a function f of z inside the circle of convergence. Suppose there is another power series

$$\sum_{n=0}^{\infty} b_n z^n$$

whose sum agrees with f in some neighborhood of the origin, then $a_n = b_n$ for $n = 0,1,2,\ldots$; that is, there is a unique power series which converges to the function f in a neighborhood of the origin.

In the case of Dirichlet series there is the following analogous theorem:

A Generalization of the Uniqueness Theorem to Dirichlet Series.

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series which converges to a function f in the right half-plane of convergence. Suppose there is another Dirichlet series,

$$\sum_{n=1}^{\infty} b_n e^{-\mu_n z},$$

whose sum agrees with the function f in some right half-plane, then $a_n = b_n$ and $\lambda_n = \mu_n$ for $n=1,2,\ldots$; that is, there is a unique Dirichlet series which has the sum $f(z)$ in a right half-plane.

Since we have uniqueness of the sum function in the region of convergence for both the power series and the Dirichlet series, then we would expect that the coefficients of the series have some relation to the sum function. This is indeed the case for both the power series and the Dirichlet series as established in the following theorems:

Cauchy or Taylor Formula for Coefficients.

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1.

Let f be the function of z to which the series converges. Then the function f is analytic inside the unit circle and the coefficients a_n are given by the following formula:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

A Generalization of Cauchy's Formula for Coefficients to Dirichlet Series [25, p.170].

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of absolute convergence $A < +\infty$. Let f be the function of z to which the series converges. Then the function f is analytic in the right half-plane $\mathrm{Re}(z) > A$ and the coefficients a_n and exponents λ_n satisfy the following condition:

$$a_n e^{-\lambda_n \sigma_1} = \lim_{T \to +\infty} \int_{t_0}^{T} f(\sigma_1 + ti) e^{\lambda_n ti}\, dt$$

where t_0 is arbitrary, where $\sigma_1 > A$, otherwise arbitrary, and where $f(\sigma_1 + ti)$ is the value of the principal branch of the function.

Outside of the circle of convergence, we saw by the Cauchy-Hadamard theorem that a power series diverges. However, the analytic function to which the series converges may exist outside of the circle of convergence. Divergence of a power series both on the circle of convergence and outside it usually occurs because of oscillation of the series. Averaging methods of Abel, Cesáro, Hölder, Borel, and others have given us sums of power series in some regions of divergence.

Cesáro's method of summing a power series is the following [27,p.309]:

Given a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

for all values of z for which the following limit exists, the series is said to be Cesáro summable (C,1):

$$C(z) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} a_n z^n}{n}$$

For the values of z for which the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges we have $C(z) = f(z)$ where f is the function to which the series converges.

Since power series diverge so crudely outside the circle of convergence, the Cesáro method of summation will apply only on and inside the circle of convergence and stronger methods are needed to sum a power series outside of the circle of convergence.

As we have seen, convergence behavior of the general Dirichlet series can be so delicate that the half-planes of the various convergences can vary greatly, even over the entire plane, whereas for a power series, all convergence is abruptly cut off at the circle of convergence.

Therefore, summation of the Cesáro type which only applies on and inside the circle of convergence in the case of power series may have a greater applicability in the case of Dirichlet series.

Marcel Riesz [17, p.21], [21], [30] introduced the following method of summation for Dirichlet series which is a generalization of Cesáro's method of summation for power series:

Given the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z},$$

let

$$S_K = \limsup_{\omega\to+\infty} \frac{\ln\left|\sum_{\lambda_n<\omega} a_n e^{(\lambda_n^2-\omega^2)} (\omega-\lambda_n)^K\right|}{\omega}$$

where K is any positive constant. If S_K is nonnegative, it may also be obtained by the formula

$$S_K = \limsup_{n\to+\infty} \frac{\ln\left|\sum_{\lambda_n<\omega} a_n (\omega-\lambda_n)^K\right|}{\omega},$$

and if S_K is nonpositive, it may be obtained by the formula

$$S_K = \limsup_{n\to+\infty} \frac{\ln\left|\sum_{\lambda_n<\omega} a_n(\lambda_n-\omega)^K\right|}{\omega} .$$

Let T denote the vertical line with abscissa S_K. Let R(T) denote the right half-plane $Re(z) > S_K$ and L(T) denote the left half-plane $Re(z) < S_K$. Then there are three possibilities:

1. If $S_K = +\infty$, the following limit will not exist for any finite z;
2. If $-\infty < S_K < +\infty$, the following limit will exist for all $z \varepsilon R(T)$ and will not exist for all $z \varepsilon L(T)$;
3. If $S_K = -\infty$, the following limit will exist for all finite z.

$$C(z) = \lim_{\omega\to+\infty} \frac{\sum_{\lambda_n<\omega} (\omega-\lambda_n)^K a_n e^{-\lambda_n z}}{\omega^K}$$

In the right half-plane where the limit exists, the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

is said to be Riesz summable (λ, K).

In the right half-plane of convergence we have $C(z) = f(z)$ where f is the function to which the series converges.

Now, we will note that the various abscissas obey the following inequalities:

$$-\infty \leq S_K \leq C \leq U \leq A \leq +\infty \quad [21]$$

where, again, equality cannot hold in every instance.

Inside the circle of convergence of a power series, the series converges to an analytic function f. By summation methods we have formed continuations of the function f onto and beyond the circle of convergence. To facilitate investigation of such continuation, it is convenient to introduce the concept of regular points and singular points. A regular point r is a point for which there can be found a neighborhood N(r) and an analytic function ϕ_r such that $\phi_r(z) = f(z)$ for all values z common to N(r) and the interior of the circle of convergence. A singular point is a point which is not a regular point.

Investigations of such continuations of Dirichlet series also involve the concept of regular points and singular points. Since the region of convergence of a Dirichlet series is a right half-plane instead of the interior of a circle for a power series, then in the definition of a regular point r we require that there be a neighborhood N(r) and an analytic function ϕ_r such that $\phi_r(z) = f(z)$ for all values z common to N(r) and the right half-plane of convergence of the Dirichlet series to the function f. In the case of Dirichlet series a singular point is still a point which is not a regular point.

It is well to notice that one essential difference between the region of convergence of a power series and that of a Dirichlet series is that the circle of convergence of a power series passes through the singular point nearest the center, whereas for a Dirichlet series there is not necessarily any singular point on the line of convergence [11, p.107].

CHAPTER III

EFFECT OF THE COEFFICIENTS ON THE CONTINUABILITY OF THE SERIES ACROSS THE BOUNDARY OF CONVERGENCE

Cauchy's formula

$$a_n = \frac{f^{(n)}(0)}{n!}$$

for the coefficients of a power series

$$\sum_{n=0}^{\infty} a_n z^n$$

shows a relationship between the coefficients of the series and the function f to which it converges. The singular points of the function f are related to the coefficients of the series, and this relationship has been a topic of investigation. The coefficients of a Dirichlet series are related to the sum function in a manner similar to the relationship for power series as shown in the following, corresponding theorems:

<u>Pringheim's Theorem [26, p.389], [29].</u>

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. If the coefficients a_n are real and nonnegative, then $z = 1$ is a singular point of f.

<u>Landau's Generalization of Pringheim's Theorem to Dirichlet Series [3, p. 79], [22].</u>

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence $C = 0$. Let f be the function of z to which the series converges. If the coefficients a_n are real and nonnegative, then $z = 0$ is a singular point of f.

Proof:

Let x_0 be a positive real number. The function f will be analytic in a neighborhood of x_0 and we can represent f by the power series expansion:

$$(1) \qquad f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (z-x_0)^k.$$

This formula is valid on the interior of the largest circle with center x_0 in which f is analytic. The point x_0 is inside the right half-plane of convergence of the Dirichlet series; thus we can substitute for the derivatives $f^{(k)}(x_0)$ their representation in Dirichlet series:

(2) $$f^{(k)}(x_0) = (-1)^k \sum_{n=1}^{\infty} a_n \lambda_n^k e^{-\lambda_n x_0}.$$

This gives us the expression:

(3) $$f(z) = \sum_{k=0}^{\infty} \frac{(x_0-z)^k}{k!} \sum_{n=1}^{\infty} a_n \lambda_n^k e^{-\lambda_n x_0}$$

which is valid on the interior of the largest circle with center x_0 in which f is analytic. Now let x_1 be a real number inside this circle of convergence, such that $x_1 < x_0$. If we put x_1 in place of z in the series (3), we see that all of the terms of the series are real and nonnegative. Therefore, we can invert the order of summation, and we have the following:

(4) $$f(x_1) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x_0} \sum_{k=0}^{\infty} \frac{(x_0-x_1)^k \lambda_n^k}{k!} .$$

Since

$$\sum_{k=0}^{\infty} \frac{(x_0-x_1)^k}{k!} \lambda_n^k = e^{\lambda_n(x_0-x_1)} = e^{\lambda_n x_0} \cdot e^{-\lambda_n x_1} ,$$

then

$$f(x_1) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x_0} \cdot e^{\lambda_n x_0} \cdot e^{-\lambda_n x_1} ,$$

$$f(x_1) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x_1}. \tag{5}$$

Since the double series (4) converges, then the series (5) must converge. Therefore, it must be that $x_1 > 0$. Now x_1 is subject only to the conditions of being real, $x_1 < x_0$, and being within the circle of convergence of the series (1). Therefore, the circle of convergence of (1) does not contain any negative real values. Since in the half-plane $Re(z) > 0$ f is analytic, then the circle of convergence of the series (1) is exactly $|z-x_0| = x_0$; that is, tangent to the line of convergence of the Dirichlet series at the point $z = 0$. The only point on the circumference of this circle beyond which f cannot be continued analytically is the point $z = 0$. Therefore, $z = 0$ is a singular point of the function f.

Pringheim's theorem and its generalization to Dirichlet series are still valid in the case of nonpositive real coefficients a_n and in the case of real coefficients with a finite number of sign changes. For the case of real coefficients with an infinite number of sign changes we have the following theorem and its generalization:

Beke's Theorem [35, p.105], [2].

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Suppose the coefficients a_n are real, $a_0 \geq 0$, and the sequence (a_n) has infinitely many changes in sign. Let the sequence (q_n) be the sequence of indices at changes in sign of the coefficients a_n; that is,

$$a_0 \geq 0, \ldots, a_{q_1} \geq 0,\ a_{q_1+1} < 0,\ a_{q_1+2} \leq 0, \ldots,\ a_{q_2} \leq 0,$$

$$a_{q_2+1} > 0,\ a_{q_2+2} \geq 0, \ldots .$$

Suppose the sequence (q_n) is such that

$$\sum_{n=2}^{\infty} \frac{1}{q_n}$$

converges. Then $z = 1$ is a singular point of f.

<u>A Generalization of Beke's Theorem to Dirichlet Series [35, p. 104].</u>

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence $C = 0$ such that

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0.$$

Let f be the function of z to which the series converges. Suppose the coefficients a_n are real, $a_1 \geq 0$, and the sequence (a_n) has infinitely many changes in sign. Let the sequence (q_n) be the sequence of indices at changes in sign of the coefficients a_n; that is,

$$a_1 \geq 0, \ldots, a_{q_1} \geq 0, a_{q_1+1} < 0, a_{q_1+2} \leq 0, \ldots,$$

$$a_{q_2} \leq 0, a_{q_2+1} > 0, a_{q_2+2} \geq 0, \ldots$$

Suppose that

$$\lim_{n\to\infty} \frac{n}{\lambda_{q_n}} = 0$$

and that there exists a positive number L such that

$$\lambda_{q_{n+1}} - \lambda_{q_n} > L \quad (n=1,2,\ldots).$$

Then $z = 0$ is a singular point of f.

Proof:

For the proof of this theorem we will need the following lemmas:

<u>Lemma 1 [3, p.46]</u>.

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence $C = 0$. Let f be the function of z to which the series converges. Suppose that the exponents λ_n are such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 0.$$

Let g be an entire function of z with power series expansion

$$g(z) = \sum_{n=0}^{\infty} g_n z^n.$$

Suppose that for every $\varepsilon > 0$ given, there exists an R_ε such that

$$(6) \qquad |g(z)| < e^{\varepsilon|z|} \quad \text{for} \quad |z| > R_\varepsilon.$$

Let

$$(7) \qquad \sum_{n=1}^{\infty} a_n g(\lambda_n) e^{-\lambda_n z}$$

be a Dirichlet series. Then the series (7) converges at least for $Re(z) > 0$. Let h be the function of z to which the series converges. Then we have that h has no singular points other than those of f.

Proof of Lemma 1:

We have already noted that

$$0 \le A - C \le \limsup_{n\to+\infty} \frac{\ln n}{\lambda_n} .$$

Since we have the hypothesis that

$$\lim_{n\to+\infty} \frac{n}{\lambda_n} = 0,$$

then

$$0 \leq A - C \leq \limsup_{n\to+\infty} \frac{\ln n}{\lambda_n} \leq \lim_{n\to+\infty} \frac{n}{\lambda_n} = 0.$$

Therefore, $A = C$. Since we have considered the abscissa of convergence to be $C = 0$, then the abscissa of absolute convergence is also $A = 0$. Therefore, we have by the formula for the abscissa of absolute convergence

(8) $$\limsup_{n\to+\infty} \frac{\ln \sum_{k=1}^{n} |a_k|}{\lambda_n} = 0.$$

Let $\varepsilon > 0$. Then by hypothesis there is a number R_ε such that

$$|g(z)| < e^{\varepsilon|z|} \quad \text{for} \quad |z| > R_\varepsilon.$$

Hence for $\lambda_n > R_\varepsilon$ we have that

(9) $$|g(\lambda_n)| < e^{\varepsilon\lambda_n}.$$

We now form the Dirichlet series,

(10) $$\sum_{n=1}^{\infty} a_n g(\lambda_n) e^{-\lambda_n z}.$$

The abscissa of absolute convergence A' of the series (10) will satisfy the following condition:

$$A' = \limsup_{n\to\infty} \frac{\ln \sum_{k=1}^{n} |a_k g(\lambda_k)|}{\lambda_n}$$

$$= \limsup_{n\to\infty} \frac{\ln \sum_{k=1}^{n} |a_k||g(\lambda_k)|}{\lambda_n}.$$

$$= \limsup_{n\to\infty} \ln\left[\sum_{k=1}^{N} |a_k||g(\lambda_k)| + \sum_{k=N+1}^{\infty} |a_k||g(\lambda_k)|\right]$$

where N is such that $\lambda_N > R_\varepsilon$. Let $G > 1$ be such that

$$\sum_{k=1}^{N} |a_k||g(\lambda_k)| < G \sum_{k=1}^{N} |a_k| e^{\varepsilon\lambda_k}$$

Therefore,

$$A' \leq \limsup_{n\to\infty} \frac{\ln G \sum_{k=1}^{n} |a_k| e^{\varepsilon\lambda_k}}{\lambda_n}$$

$$\leq \limsup_{n\to\infty} \left[\frac{\ln G}{\lambda_n} + \frac{\ln \sum_{k=1}^{n} |a_k| e^{\varepsilon\lambda_n}}{\lambda_n}\right]$$

$$= \limsup_{n\to\infty} \frac{\ln\left[e^{\varepsilon\lambda_n} \sum_{k=1}^{n} |a_k|\right]}{\lambda_n}$$

$$= \limsup_{n\to\infty} \left[\frac{\varepsilon\lambda_n}{\lambda_n} + \frac{\ln \sum_{k=1}^{n} |a_k|}{\lambda_n}\right]$$

$$= \varepsilon + 0 \quad (\text{by } (8))$$

$$= \varepsilon.$$

Therefore, $A' \leq \varepsilon$.

Since this holds for all positive values of ε, then we conclude that $A' \leq 0$. Therefore, the Dirichlet series

$$\sum_{n=1}^{\infty} a_n g(\lambda_n) e^{-\lambda_n z}$$

converges absolutely at least for $\mathrm{Re}(z) > 0$. Hence, the series converges at least for $\mathrm{Re}(z) > 0$.

Now, let $M(r)$ denote the maximum modulus of g on the circle $|z| = r$. Thus, by Cauchy's inequality

(11) $$|g_n| = \left|\frac{g^{(n)}(0)}{n!}\right| \leq \frac{1}{r^n} M(r).$$

Let $\varepsilon > 0$. From (6) we have that there is a number N_ε such that

$$M(r) < N_\varepsilon e^{\varepsilon r} \tag{12}$$

for all r.

From (11) and (12) we obtain

$$|g_n| \le \inf_{0<r<\infty} \frac{M(r)}{r^n}$$

$$\le \inf_{0<r<\infty} \frac{N_\varepsilon e^{\varepsilon r}}{r^n}$$

$$= N_\varepsilon \min_{0<r<\infty} \frac{e^{\varepsilon r}}{r^n} .$$

For the minimum point we have

$$\frac{d}{dr}\left(\frac{e^{\varepsilon r}}{r^n}\right) = \frac{r^n \varepsilon e^{\varepsilon r} - e^{\varepsilon r} n r^{n-1}}{r^{2n}} = 0.$$

Therefore, $r = \frac{n}{\varepsilon}$.

This means that

$$|g_n| \le N_\varepsilon \frac{e^{\varepsilon \cdot \frac{n}{\varepsilon}}}{\left(\frac{n}{\varepsilon}\right)^n}$$

$$= N_\varepsilon \frac{\varepsilon^n e^n}{n^n}$$

$$= N_\varepsilon \frac{(e\varepsilon)^n}{n^n} .$$

Therefore,

(13) $$|g_n| < N_\varepsilon \frac{(e\varepsilon)^n}{n!} .$$

Now, we write the Dirichlet series (10) in the form of a double series

(14) $$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \sum_{k=0}^{\infty} g_k \lambda_n^k$$

which satisfies the following:

$$\left| \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \sum_{k=0}^{\infty} g_k \lambda_n^k \right| \le \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n z} \right| \left| \sum_{k=0}^{\infty} g_k \lambda_n^k \right|$$

$$\le \sum_{n=1}^{\infty} |a_n e^{-\lambda_n z}| \sum_{k=0}^{\infty} |g_k| \lambda_n^k$$

$$< \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n z} \right| \sum_{k=0}^{\infty} N_\varepsilon \frac{(e\varepsilon)^k}{k!} \lambda_n^k \quad \text{(by(13))}$$

$$\le N_\varepsilon \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n z} \right| e^{e\varepsilon\lambda_n}$$

$$= N_\varepsilon \sum_{n=1}^{\infty} \left| a_n e^{-\lambda_n(z-e\varepsilon)} \right|$$

which converges for all z such that $\mathrm{Re}(z-e\varepsilon) > 0$. Since ε and hence $e\varepsilon$ can be made arbitrarily small, then this tells us that the double series (14) converges absolutely for $\mathrm{Re}(z) > 0$.

Therefore, we can invert the order of summation in (14) and obtain the following for z such that $\mathrm{Re}(z) > 0$:

$$h(z) = \sum_{k=0}^{\infty} g_k \sum_{n=1}^{\infty} a_n \lambda_n^k e^{-\lambda_n z}$$

$$= \sum_{k=0}^{\infty} (-1)^k g_k f^{(k)}(z). \tag{15}$$

Suppose now that the functions f and h have been analytically continued across the line $\mathrm{Re}(z) = 0$. Let S be the set of all singular points of f. We now construct for each point $s' \in S$ a circle with radius R having its center at the point s'. Let C_R be the set of all points on or inside at least one of these circles. Let E_R be the complement of C_R. Therefore, E_R contains all the points in the right half-plane $\mathrm{Re}(z) > R$. Let D_R be the component of E_R which contains the right half-plane $\mathrm{Re}(z) > R$. Now let z be any point in the set D_R. Connect z by a curve L, located entirely within the set D_R, to a point z_0 in the right half-plane $\mathrm{Re}(z) > R$. Let 3d designate the smallest distance between L and the boundary of D_R. Thus, f will be analytic in every circle of radius R + 2d, drawn around any point of L as center. Designate by M the maximum of the modulus of f in the region contained on and within these circles. Let D be the domain obtained by taking

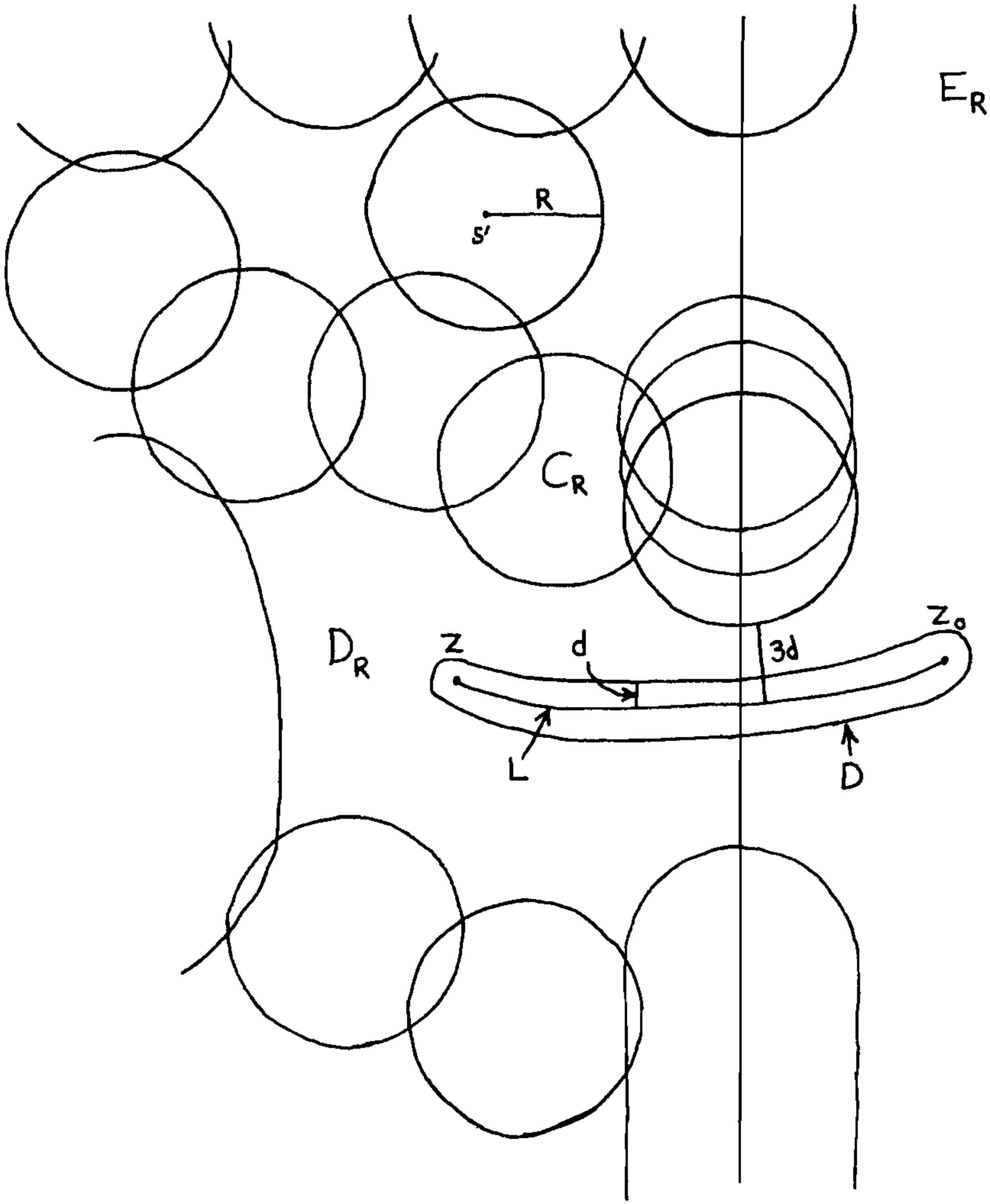

Figure 1. Construction in the Proof of Lemma 1

the union of all the circles of radius d whose center is on L. Using Cauchy's integral formula for a point in D and around a circle of radius R + d we obtain

$$\left|f^{(k)}(z)\right| \le \frac{1}{2\pi} \frac{2\pi(R+d)}{(R+d)^{k+1}} \; k!M$$

$$= \frac{Mk!}{(R+d)^k}$$

Therefore, for the series (15) we have the following:

$$\left|\sum_{k=0}^{\infty} (-1)^k g_k f^{(k)}(z)\right| \le \sum_{k=0}^{\infty} \left|g_k\right| \left|f^{(k)}(z)\right|$$

$$\le \sum_{k=0}^{\infty} N_\varepsilon \frac{\varepsilon^k}{k!} \cdot \frac{Mk!}{(R+d)^k}$$

$$(16) \qquad = \sum_{k=0}^{\infty} MN_\varepsilon \left(\frac{\varepsilon}{R+d}\right)^k .$$

For ε taken to be less than R + d, the series converges because the terms in the series are less in modulus than the terms of the geometric series (16) multiplied by a constant. Thus, the series converges absolutely and uniformly in D.

Therefore, the function h is analytic in D and, in particular, on the curve L, but this curve is by supposition any interior curve in the domain D_R.

The function h can then be continued analytically along any interior curve in the domain D_R; that is, h cannot have singular points in the interior of D_R. Therefore, the function h can have no singular points other than those of the function f. $\square$

Lemma 2 [23, p.80], [8, p.185,186].

Let (r_n) be a sequence of strictly increasing positive real numbers such that

$$\lim_{n\to\infty} \frac{n}{r_n} = 0$$

and there exists a positive number L such that

$$r_{n+1} - r_n > L \quad (n=1,2,\ldots).$$

Let

(17) $$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{r_n^2}\right)$$

be an infinite product. This infinite product defines an entire function g of z such that for any positive number $\varepsilon > 0$ given there exists an R_ε such that

(18) $$|g(z)| < e^{\varepsilon|z|} \quad \text{for } |z| > R_\varepsilon.$$

and there exists an M_ε such that

$$(19) \qquad |g(z)| < e^{-\varepsilon|z|}$$

for $|z| > M_\varepsilon$ and $|z-r_n| > \frac{L}{2}$.

<u>Proof of Lemma 2:</u>

Since

$$\lim_{n\to\infty} \frac{n}{r_n} = 0,$$

then the series

$$\sum_{n=1}^{\infty} \frac{z^2}{r_n^2}$$

converges for all z and uniformly on compact subsets. Therefore, g is an entire function by the properties of infinite products.

To prove (18), we do the following:

Since

$$\lim_{n\to\infty} \frac{n}{r_n} = 0,$$

then for every $\varepsilon > 0$ given, there exists an integer N_ε such that

$$\frac{1}{r_n} < \frac{\varepsilon}{4\pi n} \quad \text{for } n > N_\varepsilon.$$

Now

$$|g(z)| \le \prod_{n=1}^{N_\varepsilon} \left(1 + \frac{|z|^2}{r_n^2}\right) \prod_{n=N_\varepsilon+1}^{\infty} \left(1 + \frac{\varepsilon^2|z|^2}{16\pi^2 n^2}\right) .$$

For the two factors we obtain

$$\prod_{n=1}^{N_\varepsilon} \left(1 + \frac{|z|^2}{r_n^2}\right) < Pe^{\frac{\varepsilon}{4}|z|} \quad \text{for } |z| > K_\varepsilon \tag{20}$$

where P and K_ε are constants, since a polynomial is dominated by the exponential; and

$$\prod_{n=N_\varepsilon+1}^{\infty} \left(1 + \frac{\varepsilon^2|z|^2}{16\pi^2 n^2}\right) \le \prod_{n=1}^{\infty} \left(1 + \frac{\varepsilon^2|z|^2}{16\pi^2 n^2}\right)$$

$$= \frac{\sin \frac{i\varepsilon|z|}{4}}{\frac{i\varepsilon|z|}{4}}$$

$$< e^{\frac{\varepsilon}{4}|z|} \quad \text{for } |z| > 1. \tag{21}$$

The last inequality can be seen to be valid from the power series expansions of the sine and exponential functions. Thus, by (20) and (21) we have

$$|g(z)| < Pe^{\frac{\varepsilon}{2}|z|} \quad \text{for } |z| > \max(K_\varepsilon, 1)$$

Therefore,

$$|g(z)| < e^{\varepsilon|z|} \quad \text{for } |z| > R_\varepsilon$$

where $R_\varepsilon = \max\left\{K_\varepsilon, 1, P^{\frac{2}{\varepsilon}}\right\}$.

To prove (19) we first note that the inequality holds when z belongs to the sector

$$\frac{\pi}{4} < |\arg z| < \frac{3\pi}{4} .$$

In fact, the real part of z^2 is then negative so that all of the factors of the infinite product (16) are greater than 1 in modulus. Therefore, (16) is then greater than 1 in modulus and the inequality (18) is necessarily valid.

It is then sufficient to consider the values of z in the sectors

$$|\arg z| \leq \frac{\pi}{4} \quad \text{and} \quad |\pi - \arg z| \leq \frac{\pi}{4} ,$$

and since these sectors are symmetric and (16) is an even function, then it is sufficient to consider only the sector

(22) $$|\arg z| \leq \frac{\pi}{4} .$$

Let $z = x + y_i$ be in the sector (22) and exterior to all circles $|z-r_n| \leq \frac{L}{2}$.

We now determine the index m by the condition

$$r_{m-1} < x \leq r_m.$$

With these conditions we have

$$|z-r_{m-1}| > \frac{L}{2}, \quad |z-r_m| > \frac{L}{2}$$

and

$$|z-r_n| \geq |x-r_n|.$$

Therefore, we can write

$$\frac{1}{|g(z)|} < \frac{4}{L^2} \prod_{n=1}^{m-2} \frac{1}{\frac{x^2}{r_n^2} - 1} \cdot \prod_{n=m+1}^{\infty} \left(\frac{1}{1 - \frac{x^2}{r_n^2}} \right)$$

$$\leq \frac{4}{L^2} \prod_{n=1}^{m-2} \frac{r_n^2}{x^2 - r_n^2} \prod_{n=m+1}^{\infty} \left(1 + \frac{x^2}{(r_n+x)(r_n-x)} \right).$$

Let

$$\mu_n = \frac{r_n}{L}$$

and

$$\xi = \frac{x}{L} .$$

Now we have

$$\mu_{n+1} - \mu_n > 1$$

and

$$\mu_{m-1} < \xi \le \mu_m .$$

Also, let

$$\eta_n = \max_{k \ge n} \frac{k}{\mu_k} .$$

Now we can write

$$\frac{1}{|g(z)|} < \frac{4}{L^2} \prod_{n=1}^{m-2} \frac{\mu_n^2}{\xi^2 - \mu_n^2} \prod_{n=m+1}^{\infty} \left(1 + \frac{\xi^2}{(\mu_n + \xi)(\mu_n - \xi)} \right) .$$

$$= \frac{4}{L^2} \cdot P(\xi) \cdot Q(\xi)$$

where

$$P(\xi) = \prod_{n=1}^{m-2} \frac{\mu_n^2}{\xi^2 - \mu_n^2}$$

and

$$Q(\xi) = \prod_{n=m+1}^{\infty} \left\{1 + \frac{\xi^2}{(\mu_n+\xi)(\mu_n-\xi)}\right\}$$

Now

$$P(\xi) = \prod_{n=1}^{m-2} \frac{\mu_n^2}{\xi^2 - \mu_n^2}$$

$$\leq \prod_{n=1}^{m-2} \frac{\mu_m - 2}{\xi - \mu_n}$$

$$\leq \frac{\mu_{m-2}^{m-2}}{(m-2)!}$$

$$\leq \left(\frac{\mu_{m-2}e}{m-2}\right)^{m-2}$$

$$\leq \left(\frac{e}{\eta_{m-2}}\right)^{\mu_{m-2}\eta_{m-2}}$$

$$< e^{\xi\eta_{m-2}\ln \frac{e}{\eta_{m-2}}} .$$

Since

$$\lim_{n\to\infty} \eta_n = 0,$$

then

$$\lim_{m\to\infty} \eta_{m-2} \ln \frac{e}{\eta_{m-2}} = 0$$

and there is a positive number W_ε such that

$$P(\xi) < e^{\xi L \frac{\varepsilon}{8}}$$

for

$$\xi > \frac{W_\varepsilon}{L} .$$

Now

$$Q(\xi) = \prod_{n=m+1}^{\infty} \left[1 + \frac{\xi^2}{(\mu_n+\xi)(\mu_n-\xi)}\right]$$

$$\leq \prod_{n=m+1}^{\infty} \left[1 + \frac{\xi^2}{\mu_n(n-m)}\right]$$

$$\leq \prod_{n=m+1}^{\infty} \left[1 + \frac{(\xi\sqrt{\eta_m})^2}{n(n-m)}\right]$$

$$\leq \prod_{k=1}^{\infty} \left[1 + \frac{(\xi\sqrt{\eta_m})^2}{k^2}\right]$$

$$= \frac{\sin \pi i\ \xi\sqrt{\eta_m}}{\pi i\ \xi\sqrt{\eta_m}} .$$

Therefore, from the series expansions of the sine and exponential functions and since

$$\lim_{n\to\infty} \eta_n = 0,$$

then there is a positive number S_ε such that

$$Q(\xi) < e^{\xi L \frac{\varepsilon}{8}}$$

for

$$\xi > \frac{S_\varepsilon}{L} .$$

Therefore,

$$\begin{aligned}\frac{1}{|g(z)|} &< \frac{4}{L^2} \cdot e^{\xi L \frac{\varepsilon}{8}} \cdot e^{\xi L \frac{\varepsilon}{8}} \\ &= \frac{4}{L^2} e^{\xi L \frac{\varepsilon}{4}} \\ &= \frac{4}{L^2} e^{\frac{\varepsilon}{4} x} \\ &< \frac{4}{L^2} e^{\frac{\varepsilon}{2}|z|}\end{aligned}$$

for $|z| > \max(W_\varepsilon, S_\varepsilon)$.

Therefore,

$$\frac{1}{|g(z)|} < e^{\varepsilon|z|} \tag{23}$$

for $|z| > M_\varepsilon$ and $|z-r_n| > \frac{L}{2}$ where $M_\varepsilon = \max\left(W_\varepsilon, S_\varepsilon, \left(\frac{4}{L^2}\right)^{\frac{2}{\varepsilon}}\right)$. Inequality (23) can be rewritten to give (19):

$$|g(z)| > e^{-\varepsilon|z|} . \quad \square$$

Lemma 3 [4,p.13].

Let g be an entire function such that for every $\varepsilon > 0$ the following condition holds:

(24) there exists on R_ε such that

$$|g(z)| < e^{\varepsilon|z|} \quad \text{for } |z| > R_\varepsilon,$$

then for g' the following condition holds: there exists a K_ε such that

$$|g'(z)| < e^{\varepsilon|z|} \quad \text{for } |z| > K_\varepsilon.$$

Proof of Lemma 3:

Let $M(r)$ denote the maximum modulus of g on the circle $|z| = r$ and let $M_1(r)$ denote the maximum modulus of g' on the circle $|z| = r$.

Now, as a consequence of Cauchy's integral formula we have, for $|z| = r$,

$$g'(z) = \frac{1}{2\pi i} \int_{|z|=2r} \frac{g(w)}{(w-z)^2}\, dw.$$

Therefore,

$$M_1(r) \le \frac{1}{2\pi}\, 2\pi \cdot 2r \cdot \frac{M(2r)}{r^2}$$

$$< \frac{2M(2r)}{r}$$

(25) $$< 2M(2r) \quad \text{for } r > 1.$$

For $r > R_{\frac{\varepsilon}{4}}$ we have

$$M(2R) < e^{2 \cdot \frac{\varepsilon}{4} r}$$

(26) $$= e^{\frac{\varepsilon}{2} r} .$$

Therefore, from (25) and (26) we obtain

$$M_1(r) < 2e^{\frac{\varepsilon}{2} r} \quad \text{for } r > R_{\frac{\varepsilon}{4}} .$$

Therefore,

$$M_1(r) < e^{\varepsilon r} \quad \text{for } r > \max\{R_{\frac{\varepsilon}{4}}, 2^{\frac{2}{\varepsilon}}\} .$$

This means that for $K_\varepsilon = \max(R_{\frac{\varepsilon}{4}}, 2^{\frac{2}{\varepsilon}})$ we have

$$|g'(z)| < e^{\varepsilon |z|} \quad \text{for } |z| > K_\varepsilon. \quad \square$$

Lemma 4 [36, p.287].

Let (r_n) be a sequence of strictly increasing positive real numbers such that

$$\lim_{n\to+\infty} \frac{n}{r_n} = 0.$$

Let g be the entire function defined by

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{r_n^2}\right).$$

Let (λ_n) be a sequence of strictly increasing positive real numbers such that

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0$$

and there exists a positive number L such that

$$r_n - r_{n-1} > L$$

and

$$|r_n - \lambda_k| > L \qquad \text{for } (k,n=1,2,\ldots).$$

Suppose the Dirichlet series,

$$(27) \qquad \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

has abscissa of convergence $C = 0$. Then the Dirichlet series,

$$\sum_{n=1}^{\infty} a_n g(\lambda_n) e^{-\lambda_n z} \tag{28}$$

has abscissa of convergence $C' = 0$.

Proof of Lemma 4:

Since the abscissa of convergence of the series (27) is $C = 0$, we obtain the following

$$\limsup_{r\to+\infty} \frac{\ln\left|\sum_{[r]\le\lambda_n<r} a_n\right|}{r} = 0. \tag{29}$$

For the abscissa of convergence C' on the series (28) we have

$$C' = \limsup_{r\to+\infty} \frac{\ln\left|\sum_{[r]\le\lambda_n r} a_n g(\lambda_n)\right|}{r}. \tag{30}$$

From (29) we obtain that for any $\varepsilon > 0$ there is a positive constant N_ε such that

$$\left|\sum_{[r]\le\lambda_n<r} a_n\right| < e^{\frac{\varepsilon}{2} r}$$

for all r such that $[r] > N_\varepsilon$. Therefore,

$$\left|\sum_{[r]\le\lambda_n<r} a_n\right| < e^{\varepsilon[r]}$$

for all r such that $[r] > N_\varepsilon$.

Let n_1 and n_2 be the indices such that

$$\lambda_{n_1-1} < [r],\ \lambda_{n_1} \ge [r],\ \lambda_{n_2} < r,\ \lambda_{n_2+1} \ge r.$$

Now, by Abel's transformation we have

$$\sum_{[r]\le\lambda_n<r} a_n g(\lambda_n) = \sum_{n=n_1}^{n_2} a_n g(\lambda_n)$$

$$= \sum_{n=n_1}^{n_2-1} \left[g(\lambda_n)-g(\lambda_{n+1})\right]\left(\sum_{k=n_1}^{n} a_k\right) + g(\lambda_{n_2})\left(\sum_{k-n_1}^{n_2} a_k\right) .$$

Now, by Lemma 2 we have

$$\left|\sum_{[r]\le\lambda_n<r} a_n g(\lambda_n)\right| \le e^{\varepsilon[r]}\left[\sum_{n=n_1}^{n_2-1} \int_{\lambda_n}^{\lambda_{n+1}} |g'(t)|dt + |g(\lambda_{n_2})|\right]$$

$$< e^{\varepsilon[r]}\left[\int_{[r]}^{r} |g'(t)|dt + e^{\varepsilon r}\right] \tag{31}$$

for $[r] > \max(R_\varepsilon, N_\varepsilon)$

where R_ε is as in Lemma 2.

By Lemma 3 there exists a positive constant K_ε such that

$$|g'(r)| < e^{\varepsilon r} \quad \text{for } r > K_\varepsilon. \tag{32}$$

Therefore, from (31) and (32) we obtain

$$\left|\sum_{[r]\le\lambda_n<r} a_n g(\lambda_n)\right| < 2e^{\varepsilon[r]+\varepsilon r}$$

$$\le 2e^{2\varepsilon r} .$$

Thus, by (30) we have

$$C' \le \limsup_{r\to+\infty} \frac{\ln(2e^{2\varepsilon r})}{r}$$

$$= \limsup_{r\to+\infty}\left[\frac{\ln 2}{r} + \frac{\ln e^{2\varepsilon r}}{r}\right]$$

$$= \limsup_{r\to+\infty}\left[0 + \frac{2\varepsilon r}{r}\right]$$

$$= \limsup_{r\to+\infty} \quad 2\varepsilon$$

$$= 2\varepsilon.$$

Since this holds for all $\varepsilon > 0$, then we have

$$C' \leq 0.$$

By (30), for any given $\varepsilon > 0$, we obtain

$$\left| \sum_{[r] \leq \lambda_n < r} a_n g(\lambda_n) \right| < e^{(C'+\varepsilon)[r]} \quad \text{for } [r] > T_\varepsilon \tag{33}$$

where T_ε is some positive constant.

Now by again using Abel's transformation we obtain the following equations:

$$\sum_{[r] \leq \lambda_n < r} a_n = \sum_{n=n_1}^{n_2} a_n g(\lambda_n) \cdot \frac{1}{g(\lambda_n)}$$

$$= \sum_{n=n_1}^{n_2-1} \left(\frac{1}{g(\lambda_n)} - \frac{1}{g(\lambda_{n+1})} \right) \cdot \left(\sum_{k=n_1}^{n} a_k g(\lambda_k) \right)$$

$$+ \frac{1}{g(\lambda_{n_2})} \cdot \left(\sum_{k=n_1}^{n_2} a_k g(\;_k) \right),$$

$$= \sum_{n=n_1}^{n_2-1} \left[\left(\frac{g(\lambda_{n+1}) - g(\lambda_n)}{g(\lambda_n)g(\lambda_{n+1})} \right) \cdot \left(\sum_{k=n_1}^{n} a_k g(\lambda_k) \right) \right]$$

$$+ \frac{1}{g(\lambda_{n_2})} \cdot \left(\sum_{k=n_1}^{n_2} a_k g(\lambda_k) \right) .$$

We note that by hypothesis we have

$$|r_n - \lambda_k| > L \quad \text{for } (k,n=1,2,\ldots),$$

therefore

$$g(\lambda_n) \neq 0$$

for all n.

Now

$$\left| \sum_{[r]\le\lambda_n<r} a_n \right| \le \sum_{n=n_1}^{n_2-1} \frac{|g(\lambda_{n+1}) - g(\lambda_n)|}{|g(\lambda_n)g(\lambda_{n+1})|} \cdot \left| \sum_{k=n_1}^{n} a_k g(\lambda_k) \right|$$

$$+ \frac{1}{|g(\lambda_{n_2})|} \left| \sum_{k=n_1}^{n_2} a_k g(\lambda_k) \right|$$

$$\leq e^{(C'+\varepsilon)[r]}\left[\sum_{n=n_1}^{n_2-1}\frac{\int_{\lambda_n}^{\lambda_{n+1}}|g'(t)|dt}{|g(\lambda_n)||g(\lambda_{n+1})|}+\frac{1}{|g(\lambda_{n_2})|}\right]$$

by (33). Now by hypothesis we have

$$|r_n-\lambda_k| > L \qquad \text{for } (k,n=1,2,\ldots).$$

Therefore by Lemma 2 there exists a positive number $M_\varepsilon > T_\varepsilon$ such that for $[r] > M_\varepsilon$ we have

$$\left|\sum_{[r]\leq\lambda_n<r} a_n\right| \leq e^{(C'+\varepsilon)[r]}\left[e^{2\varepsilon r}\int_{[r]}^{r}|g'(t)|dt + e^{\varepsilon r}\right]$$

$$< 2e^{(C'+\varepsilon)[r]+3\varepsilon r} \quad \text{(by Lemma 3)}$$

(34) $$\leq 2e^{C'r+4\varepsilon r} \qquad \text{for } [r] > M_\varepsilon.$$

Therefore, by (29) and (34) we have

$$0 = C \leq \limsup_{r\to+\infty}\frac{\ln(2e^{C'r+4\varepsilon r})}{r}$$

$$= \limsup_{r\to+\infty}\left[\frac{\ln 2}{r}+\frac{\ln e^{C'r+4\varepsilon r}}{r}\right]$$

$$= \limsup_{r\to+\infty} \left[0 + \frac{C'r + 4\varepsilon r}{r} \right]$$

$$= \limsup_{r\to+\infty} \left[C' + 4\varepsilon \right]$$

$$= C' + 4\varepsilon.$$

Therefore,

$$0 = C \leq C' + 4\varepsilon.$$

Again since this holds for all $\varepsilon > 0$, then we have

$$0 \leq C'.$$

We have now shown that $C' \leq 0$ and $C' \geq 0$. Therefore $C' = 0$. ▯

Now we return to the proof of the generalization of Beke's theorem to Dirichlet series.

Let g be a function of z defined by the formula

$$g(z) = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(\lambda_{q_n} + \lambda_{q_n+1})^2} \right].$$

Let $r_n = \frac{1}{2}(\lambda_{q_n} + \lambda_{q_n+1})$.

Now we have

$$g(z) = \prod_{n=1}^{\infty} \left[1 - \frac{z^2}{r_n^2}\right].$$

The hypotheses of Lemma 4 are satisfied. We can see this by noting that the λ_n's are separated by hypothesis and the definition of r_n shows that the r_n's have the same separation and are separated from the λ_n's by at least half of this separation. Also, by the definition of r_n and the hypothesis

$$\lim_{n\to\infty} \frac{n}{\lambda_{q_n}} = 0$$

then it follows that

$$\lim_{n\to\infty} \frac{n}{r_n} = 0 \ .$$

Therefore, by Lemma 4 we have that the Dirichlet series

$$\sum_{n=1}^{\infty} a_n g(\lambda_n) e^{-\lambda_n z} \tag{35}$$

also has abscissa of convergence $C' = 0$. Let h be the function of z to which the series converges.

The hypotheses of Lemma 1 are satisfied, since by Lemma 2 g satisfies the required growth condition. Therefore, the function h has no singular points other than those of the function f.

Now we have that $\lambda_{q_n} < r_n < \lambda_{q_n+1}$. Therefore, by considering the definition of g, we see that

$$g(\lambda_1) > 0, \ldots, g(\lambda_{q_1}) > 0, \; g(\lambda_{q_1+1}) < 0, \ldots$$

$$\ldots, g(\lambda_{q_2}) < 0, \; g(\lambda_{q_2+1}) > 0, \ldots .$$

Therefore, the Dirichlet series (31) has all nonnegative coefficients. Thus, by Landau's generalization of Pringheim's theorem, the point z = 0 is a singular point for the function h. Therefore, by Lemma 1 it must also be a singular point of the function f. ▯

We now generalize the situation further with the following theorem:

<u>Fabry's Theorem</u>.

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Define $b_n = \mathrm{Re}(a_n)$ and $c_n = \mathrm{Im}(a_n)$. At least one of the series

$$\sum_{n=0}^{\infty} b_n z^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} c_n z^n$$

has radius of convergence 1, and if that series satisfies the hypotheses of Beke's theorem, then $z = 0$ is a singular point of f.

A Generalization of Fabry's Theorem to Dirichlet Series.

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \tag{36}$$

be a Dirichlet series with abscissa of convergence $C = 0$. Let f be the function of z to which the series converges. Define $b_n = \mathrm{Re}(a_n)$ and $c_n = \mathrm{Im}(a_n)$. At least one of the Dirichlet series

$$\sum_{n=1}^{\infty} b_n e^{-\lambda_n z} \quad \text{and} \quad \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} \tag{37}$$

has abscissa of convergence $c = 0$, and if that series satisfies the generalization of Beke's theorem to Dirichlet series, then $z = 0$ is a singular point of f.

Proof:

Since for real values of z the series (37) represent the real and imaginary parts, respectively, of the series (36) which converges for $\mathrm{Re}(z) > 0$, then the series (37) each converge at least for $\mathrm{Re}(z) > 0$. Let f_1 and f_2 be, respectively, the functions of z to which the series converge. The functions f, f_1, and f_2 are analytic for $\mathrm{Re}(z) > 0$, thus all points such that $\mathrm{Re}(z) > 0$ are regular points of the functions f, f_1, and f_2. We want to prove that if at least one of the functions

f_1 and f_2 has $z = 0$ as a singular point, then f has $z = 0$ as a singular point. We will prove that the fact that $z = 0$ is a regular point of f will require $z = 0$ to be a regular point of f_1 and f_2.

If $z = 0$ is a regular point of f_1, then f has a power series expansion at $z = 0$. Let

$$\sum_{n=0}^{\infty} d_n z^n$$

be the expansion, and let $r > 0$ be its radius of convergence. Therefore, for $|z| < r$ we have

$$f(z) = \sum_{n=0}^{\infty} d_n z^n. \tag{38}$$

Let $p_n = \mathrm{Re}(d_n)$ and $q_n = \mathrm{Im}(d_n)$. Therefore, the series

$$\sum_{n=0}^{\infty} p_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} q_n z^n$$

which form the real and imaginary parts, respectively, of (38) for real values of z will each converge at least for $|z| < r$. For real values x of z such that $0 < x < r$ we have that

$$f_1(x) = \mathrm{Re}(f(x)) = \sum_{n=0}^{\infty} p_n x^n,$$

and

$$f_2(x) = \mathrm{Im}(f(x)) = \sum_{n=0}^{\infty} q_n x^n.$$

Therefore, for all $|z| < r$ we have

$$f_1(z) = \sum_{n=0}^{\infty} p_n z^n$$

and

$$f_2(z) = \sum_{n=0}^{\infty} q_n z^n.$$

Therefore, f_1 and f_2 have $z = 0$ as a regular point. Thus, if at least one of the functions f_1 and f_2 has $z = 0$ as a singular point, then f has $z = 0$ as a singular point.

Suppose one of the functions f_1 and f_2 satisfies the hypotheses of the generalization of Beke's theorem to Dirichlet series, then $z = 0$ is a singular point for at least one of the functions f_1 and f_2. Therefore, as we have seen, $z = 0$ is a singular point of the function f.$\square$

We continue with another theorem due to Fabry and its generalization to Dirichlet series:

<u>Fabry's Gap Theorem</u>.

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Let k_n be the index of the nth nonzero coefficient. Suppose that

$$\lim_{n\to\infty} \frac{n}{k_n} = 0.$$

Then the circle of convergence is the natural boundary of the function f; that is, there is no analytic continuation of the function f across the unit circle.

<u>A Generalization of Fabry's Gap Theorem to Dirichlet Series.</u>

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence C = 0. Let f be the function of z to which the series converges. Suppose that

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 0$$

and there exists a positive number L such that

$$\lambda_{n+1} - \lambda_n > L \quad (n=1,2,\ldots).$$

Then the imaginary axis is the natural boundary of the function f; that is, there is no analytic continuation of the function f across the right half plane of convergence.

Proof:

Let t be a real number. We will show that the point $z = ti$ on the imaginary axis is a singular point of f; that is, $z = 0$ is a singular point of the function g defined by

$$g(z) = f(z+ti) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n ti} e^{-\lambda_n z}$$

$$= \sum_{n=1}^{\infty} d_n e^{-\lambda_n z}$$

where $d_n = a_n e^{-\lambda_n ti}$. Define $p_n = \mathrm{Re}(d_n)$ and $q_n = \mathrm{Im}(d_n)$. Therefore, we have

$$g(z) = \sum_{n=1}^{\infty} (p_n + q_n i) e^{-\lambda_n z}$$

This series also has abscissa of convergence $C = 0$. Therefore, each of the series

$$\sum_{n=1}^{\infty} p_n e^{-\lambda_n z} \quad \text{and} \quad \sum_{n=1}^{\infty} q_n e^{-\lambda_n z}$$

satisfies either the hypotheses of the generalization of Pringheim's theorem to Dirichlet series or the hypotheses of the generalization of Beke's theorem to Dirichlet series.

In the case of Beke's theorem, since

$$q_n \geq n$$

and

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 0,$$

then

$$\lim_{n\to\infty} \frac{n}{\lambda_{q_n}} = 0.$$

Therefore, by the generalization of Fabry's theorem to Dirichlet series, $z = 0$ is a singular point of the function g, and therefore $z = ti$ is a singular point of f. Since t is an arbitrary real number, then every point on the imaginary axis is a singular point of f. Thus, the imaginary axis is the natural boundary of the function f. $\Box$

Now for a series with finite boundary of convergence, we can change the sign of some of the coefficients and make that boundary the natural boundary of the sum function. This is shown in the following theorems:

Hurwitz-Polya Theorem.

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Then there exists a sequence (ε_n), where $\varepsilon_n = \pm 1$, such that the series,

$$\sum_{n=0}^{\infty} \varepsilon_n a_n z^n$$

has the unit circle as the natural boundary.

Szász's Generalization of the Hurwitz-Polya Theorem to Dirichlet Series.

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \tag{39}$$

be a Dirichlet series with abscissa of convergence C. Let f be the function of z to which the series converges. If

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0$$

and there exists a positive number L such that

$$\lambda_{n+1} - \lambda_n > L \quad (n=1,2,\ldots),$$

then there exists a sequence (ε_n), where $\varepsilon_n = \pm 1$, such that the series,

$$\sum_{n=1}^{\infty} a_n \varepsilon_n e^{-\lambda_n z}$$

has the line Re(z) = C as the natural boundary.

Proof:

Since we have noted that

$$0 \leq A - C \leq \limsup_{n\to+\infty} \frac{\ln n}{\lambda_n}$$

and we have the hypothesis that

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0,$$

then C = A. Therefore, there is no strip of conditional convergence. We will need the following lemma, which will give us another way to find the abscissa of convergence C = A under the hypotheses of this theorem:

Lemma 5.

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \tag{39}$$

be a Dirichlet series with the property that

$$\lim_{n\to+\infty} \frac{\ln n}{\lambda_n} = 0.$$

Then the abscissa of convergence C is equal to the abscissa of absolute convergence A and they are determined by the formula

$$C = A = \limsup_{n\to+\infty} \frac{\ln|a_n|}{\lambda_n}. \tag{40}$$

Proof of Lemma 5:

We have already discussed the fact that $C = A$. Now we will discuss the determination of A.

Case 1. If A is finite; that is, $-\infty<A<+\infty$, then for every $\delta > 0$ given, there is an integer N_δ such that for $n > N_\delta$ we have

$$\lambda_n > \frac{4}{\delta} \ln n.$$

and

$$\frac{\ln|a_n|}{\lambda_n} < A + \frac{\delta}{2}$$

which gives

$$|a_n| < e^{\lambda_n\left(A + \frac{\delta}{2}\right)}$$

Thus,

$$\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n(A+\delta)} < \sum_{n=1}^{\infty} e^{-\lambda_n \frac{\delta}{2}} < \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Therefore, the series (39) is absolutely convergent for $\mathrm{Re}(z) > A$.

From (40) we see that for any $\delta > 0$ infinitely many of the indices n are such that

$$\frac{\ln|a_n|}{\lambda_n} > A - \frac{\delta}{2}$$

which gives

$$|a_n| > e^{\lambda_n\left(A - \frac{\delta}{2}\right)}.$$

Thus

$$|a_n|e^{-\lambda_n(A-\delta)} > e^{\lambda_n \frac{\delta}{2}}$$

$$> 1 \quad \text{for infinitely many indices } n.$$

Therefore, the series (39) diverges for $\mathrm{Re}(z) < A$.

Case 2. If $A = -\infty$, then for any real number r there is an integer N_r such that for $n > N_r$

$$|a_n| < e^{\lambda_n(r-2)}.$$

Thus

$$\sum_{n=1}^{\infty} |a_n|e^{-\lambda_n r} < \sum_{n=1}^{\infty} e^{-\lambda_n 2}$$

which is convergent since $\lambda_n > n$ for n sufficiently large. Therefore, the series (39) converges for every value of z.

Case 3. If $A = +\infty$, then for any positive number p and any $\delta > 0$ there are infinitely many indices n such that

$$|a_n| > e^{\lambda_n\left(p - \frac{\delta}{2}\right)} .$$

Thus,

$$|a_n| e^{-\lambda_n(p-\delta)} > e^{\lambda_n \frac{\delta}{2}}$$

$$> 1.$$

Therefore, the series (39) diverges for every value of z. ▯

Now we return to the proof of Szász's generalization. Let (a_{k_n}) be a subsequence of (a_k) such that

$$a_{k_n} \neq 0,$$

$$\lim_{n\to\infty} \frac{\ln|a_{k_n}|}{\lambda_{k_n}} = A,$$

and

$$\lim_{n\to\infty} \frac{n}{\lambda_{k_n}} = 0.$$

We now consider the Dirichlet series

$$\sum_{n=1}^{\infty} a_{k_n} e^{-\lambda_{k_n} z} \tag{41}$$

which is a subseries of (39). By Lemma 5, the series (41) has abscissa of convergence A. Let D_k be the function of z to which this series (41) converges.

Now according to the generalization of Fabry's gap theorem, the line Re(z) = A is the natural boundary of the function D_k.

Next we add to the Dirichlet series (41) the Dirichlet series

$$\sum_{n=1}^{\infty} d_n e^{-\lambda_n z} \tag{42}$$

where the coefficients d_n are the remaining coefficients of the series (39) from the construction of the series (41). The abscissa of convergence of the series (42) is C = A. Let P be the function of z to which the series converges. Thus, we have for Re(z) > A

$$f(z) = P(z) + D_k(z).$$

Now because of the absolute convergence of the series we can partition the series (41) into a sum of infinite series

$$\sum_{n=1}^{\infty} a_{k_n} e^{-\lambda_{k_n} z} = \sum_{n=1}^{\infty} \left(\sum_{r=1}^{\infty} b_{n,r} e^{-\lambda_{n,r} z} \right)$$

in which each term $a_{k_n} e^{-\lambda_{k_n} z}$ is in one and only one of the series

$$\sum_{r=1}^{\infty} b_{n,r} e^{-\lambda_{n,r} z}.$$

Because there are subseries of (41) then by Lemma 5 we have that the abscissa of absolute convergence of each series is A and the natural boundary of each series is Re(z) = A. For Re(z) > A let Q_n be the function of z to which the nth series converges.

Now we will form a new series for Re(z) > A

$$(43) \qquad \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} = \sum_{n=1}^{\infty} d_n e^{-\lambda_n z} + \sum_{n=1}^{\infty} \varepsilon_n \sum_{r=1}^{\infty} b_{n,r} e^{-\lambda_{n,r} z}$$

in which $\varepsilon_n = \pm 1$. This is the Dirichlet series (39) except that some of the terms are multiplied by -1. Since $|c_n| = |a_n|$ then the series (43) converges absolutely for Re(z) > A. Let $D_{k,\varepsilon}$ be the function to which the series converges.

We now prove that of the series of type (43) only a countable number can be continued across the line Re(z) = A. The remaining uncountably many of the series of the type (43) have the line Re(z) = A as the natural boundary.

First, suppose series of type (43) whose sum function is $D_{k,\varepsilon}$ has regular points on the line Re(z) = A so that I_ε is the open set of regular points of $D_{k,\varepsilon}$ on the line Re(z) = A. Now suppose there is another sequence (ε_n') such that $\varepsilon_n' = -\varepsilon_n$ for infinitely many of the indices n such that for the corresponding series of type (43) the sum function $D_{k,\varepsilon'}$ has regular points on the line Re(z) = A. Let $I_{\varepsilon'}$ be the open set of regular points of $D_{k,\varepsilon'}$ on the line Re(z) = A. We will now show that $I_\varepsilon \cap I_{\varepsilon'} = \phi$. If $I_\varepsilon \cap I_{\varepsilon'} = \phi$, then the series defined by

$$D_{k,\varepsilon}(z) - D_{k,\varepsilon'}(z) = \sum_{n=1}^{\infty} (\varepsilon_n - \varepsilon_n')Q_n(z) \tag{44}$$

will have a regular point on the line Re(z) = A. By the generalization of Fabry's gap theorem to Dirichlet series we see that the series (44) has the line Re(z) = A as its natural boundary. Therefore, we have a contradiction to our assumption that the open sets I_ε and $I_{\varepsilon'}$ were di - joing. Therefore, the open sets of regular points of series of type (43) with associated sequences (ε_n'') differing from the sequence (ε_n) in infinitely many places are mutually disjoint. Since there are uncountably many such series and since one can have only a countable number of nonempty disjoint open sets on the imaginary axis, then for uncountably many series of type (43) we have $I_{\varepsilon''} = \phi$. Therefore, there exist many sequences (ε_n) such that the line Re(z) = A is the natural boundary of series of type (43). ▯

CHAPTER IV

BEHAVIOR OF THE BOUNDARY OF CONVERGENCE

The Cauchy-Hadamard theorem tells us that the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges inside the circle of convergence and diverges outside it. On the circle of convergence Γ we have the following possibilities:

1. The series may converge everywhere on Γ, as shown by the example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n$$

which converges (absolutely) for all $|z| = 1$.

2. The series may converge at some points of Γ and diverge at other points of Γ, as shown by the series

$$\sum_{n=1}^{\infty} \frac{1}{n} z^n$$

which converges for $z = -1$ and diverges for $z = 1$.

3. The series may diverge everywhere on Γ, as shown by the series

$$\sum_{n=0}^{\infty} z^n$$

which diverges for all $|z| = 1$.

The Dirichlet series corresponding to these examples possess the same properties on the line of convergence.

Much of the work in the investigation of convergence of a series on its boundary of convergence was initiated by a theorem that Niels H. Abel presented in 1826. His theorem and its generalization to Dirichlet series follow.

Abel's Theorem

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Suppose the series is convergent at z = 1 and converges to B; that is,

$$B = \sum_{n=0}^{\infty} a_n,$$

then

$$\lim_{\substack{z \to 1 \\ \frac{|1-z|}{1-|z|} \leq k}} f(z) = B$$

where k is an arbitrary number greater than 1.

A Generalization of Abel's Theorem to Dirichlet Series

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence C = 0. Let f be the function of z to which the series converges. Suppose the series is convergent at z = 0 and converges to B; that is,

$$B = \sum_{n=1}^{\infty} a_n ,$$

then

$$\lim_{\substack{z \to 0 \\ Re(z) > 0 \\ |\arg z| \leq \Theta < \frac{\pi}{2}}} f(z) = B$$

where Θ is an arbitrary number such that $0 \leq \Theta < \frac{\pi}{2}$.

Tauber sought conditions on the series which would necessitate a converse to Abel's theorem. He accomplished this in the following theorem:

Tauber's Theorem

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Suppose that

$$\lim_{n\to\infty} na_n = 0.$$

If the limit

$$\lim_{x\to 1^-} f(x) = B$$

exists, then the series

$$\sum_{n=0}^{\infty} a_n$$

converges and has sum B.

In 1908 Edmund Landau presented a generalization of Tauber's theorem to Dirichlet series.

Landau's Generalization of Tauber's Theorem to Dirichlet Series

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence C = 0. Let f be the function to which the series converges. Suppose that

$$\lim_{n\to\infty} \frac{a_n\lambda_n}{\lambda_n - \lambda_{n-1}} = 0.$$

If the limit

$$\lim_{x\to 0+} f(x) = B$$

exists, then the series

$$\sum_{n=1}^{\infty} a_n$$

converges and has sum B.

In 1909 W. Schnee generalized Landau's generalization and provided necessary and sufficient conditions for the convergence of the series,

$$\sum_{n=1}^{\infty} a_n.$$

Schnee's Generalization of Tauber's Theorem to Dirichlet Series

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence C = 0. Let f be the function to which the series converges. Suppose that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \lambda_k a_k}{\lambda_n} = 0.$$

If the limit

$$\lim_{x \to 0+} f(x) = B$$

exists, then the series

$$\sum_{n=1}^{\infty} a_n$$

converges and has sum B.

Tauber's theorem led to other theorems concerning the convergence of the series on the boundary of convergence. The following theorem and its generalization are prime examples.

Fatou-Riesz Theorem

Let

$$\sum_{n=0}^{\infty} a_n z^n$$

be a power series with radius of convergence 1. Let f be the function of z to which the series converges. Suppose that

$$\lim_{n \to \infty} a_n = 0$$

Then the series converges (in fact, uniformly) on every arc

$$z = e^{i\Theta}, \quad (\alpha \leq \Theta \leq \beta)$$

all of whose points are regular points of f.

A Generalization of the Fatou-Riesz Theorem to Dirichlet Series

Let

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

be a Dirichlet series with abscissa of convergence C = 0. Let f be the function to which the series converges. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{\lambda_n - \lambda_{n-1}} = 0 \tag{45}$$

and

$$\lambda_n - \lambda_{n-1} < K$$

where K is some positive constant. Then the series converges (in fact, uniformly) on every interval $[t_1 i, t_2 i]$, $t_1 < t_2$ on the imaginary axis, all of whose points are regular points of f.

Proof:

Let B denote the right half-plane $\mathrm{Re}(z) > 0$. Let $y = ti$, where $t_1 < t < t_2$. Suppose f is regular at such points y. Then there is a neighborhood N(y) and an analytic function ϕ_y defined on N(y) such that $\phi_y(z) = f(z)$ for $z \in N(y) \cap B$. Let D be the union of B and all of the

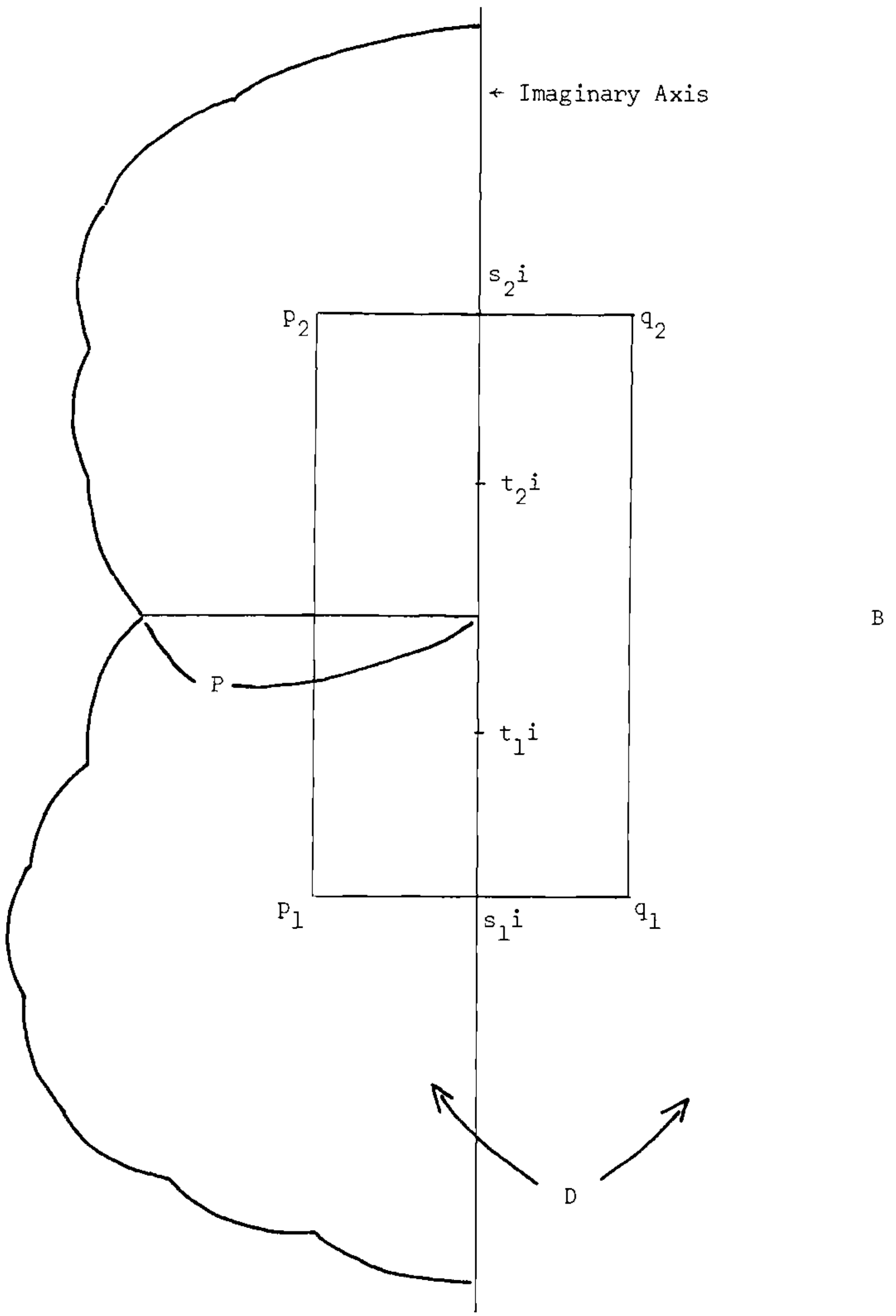

Figure 2. Construction in the Proof of the Generalization of the Fatou-Riesz Theorem to Dirichlet Series

neighborhoods $N(y)$. Then D is a domain containing both P and $[t_1 i, t_2 i]$. If we define a function ϕ on D by the following formula

$$\phi(z) = \begin{cases} f(z) & \text{if } z \in B \\ \phi_y(z) & \text{if } z \in N(y) - B \end{cases}$$

then ϕ is single valued and analytic on D. Since $\phi = f$ on B, then ϕ has the same Dirichlet series expansion on B as has f.

Now let $p > 0$ be the distance between $[t_1 i, t_2 i]$ and the boundary of D. Moreover, let $s_1 i = t_1 i - \frac{p}{2} i$ and $s_2 i = t_2 i + \frac{p}{2} i$. We now form the rectangle R with the following vertices:

$$p_1 = -\frac{p}{2} + s_1 i$$

$$p_2 = -\frac{p}{2} + s_2 i$$

$$q_1 = \frac{p}{2} + s_1 i$$

$$q_2 = \frac{p}{2} + s_2 i$$

The interior of R is a domain that contains $[t_1 i, t_2 i]$, and the rectangle and its interior are contained within D.

Let ω be a positive, real variable. We now make the following definitions:

Let g_ω and h_ω be functions defined by

$$g_\omega(z) = e^{\omega z} h_\omega(z)(z-s_1 i)(z-s_2 i)$$

where

$$h_\omega(z) = \sum_{\lambda_k \le \omega} a_k e^{-\lambda_k z} - \phi(z)$$

Note g_ω is analytic on D since $e^{\omega z}$, h_ω, $(z-s_1 i)$ and $(z-s_2 i)$ all form analytic functions on D.

We intend to estimate the function $|g_\omega(z)|$ on the perimeter of the rectangle R and then use the maximum module principle to estimate $|g_\omega(z)|$ where z is on the interior of R.

Let $\varepsilon > 0$ be given. Then by (45) there exists an integer N such that for $n > N$ we have

$$\frac{|a_n|}{\lambda_n - \lambda_{n-1}} < \varepsilon .$$

On the perimeter of R, we have the following inequality:

$$|z-s_1 i| \cdot |z-s_2 i| \le c|x| \tag{46}$$

where c is some constant and $x = \mathrm{Re}(z)$.

If z is on the perimeter of R and $\mathrm{Re}(z) > 0$, then if n is the largest index k such that $\lambda_k \le \omega$,

$$|h_\omega(z)| = \left|\sum_{k=n+1}^{\infty} a_k e^{-\lambda_k z}\right|$$

$$< \varepsilon \sum_{k=n+1}^{\infty} (\lambda_k - \lambda_{k-1}) e^{-\lambda_k x}$$

$$= \varepsilon \sum_{k=n+1}^{\infty} \left\{\lambda_k e^{-\lambda_k x} - \lambda_{k-1} e^{-\lambda_{k-1} x} + \lambda_{k-1} e^{-\lambda_{k-1} x} - \lambda_{k-1} e^{-\lambda_k x}\right\}$$

$$= \varepsilon \sum_{k=n+1}^{\infty} \left[re^{-rx} \Big|_{\lambda_{k-1}}^{\lambda_k} + \left(\lambda_{k-1} e^{-\lambda_{k-1} x} - \lambda_{k-1} e^{-\lambda_k}\right)\right]$$

$$= \varepsilon \sum_{k=n+1}^{\infty} \left[\int_{\lambda_{k-1}}^{\lambda_k} e^{-rx}\, dr - \int_{\lambda_{k-1}}^{\lambda_k} xre^{-rx}\, dr + \left(\lambda_{k-1} e^{-\lambda_{k-1} x} - \lambda_{k-1} e^{-\lambda_k x}\right)\right]$$

$$= \varepsilon \int_{\lambda_n}^{\infty} e^{-rx}\, dr + \varepsilon \sum_{k=n+1}^{\infty} \left[-x \int_{\lambda_{k-1}}^{\lambda_k} re^{-rx}\, dr + \lambda_{k-1}\left(e^{-\lambda_{k-1} x} - e^{-\lambda_k x}\right)\right]$$

$$= \varepsilon \int_{\lambda_n}^{\infty} e^{-rx}\, dr + \varepsilon \sum_{k=n+1}^{\infty} \left[-x \int_{\lambda_{k-1}}^{\lambda_k} re^{-rx}\, dr + \lambda_{k-1} x \int_{\lambda_{k-1}}^{\lambda_k} e^{-rx} dr\right]$$

$$\leq \varepsilon \int_{\lambda_n}^{\infty} e^{-rx}\, dr + \varepsilon \sum_{k=n+1}^{\infty} \left[-\lambda_{k-1} x \int_{\lambda_{k-1}}^{\lambda_k} e^{-rx}\, dr + \lambda_{k-1} x \int_{\lambda_{k-1}}^{\lambda_k} e^{-rx}\, dr \right]$$

$$= \varepsilon \int_{\lambda_n}^{\infty} e^{-rx}\, dr$$

$$= \varepsilon x^{-1} e^{-\lambda_n x} \tag{47}$$

where we remember that $\lambda_n \leq \omega < \lambda_{n+1}$. Therefore, combining (46) and (47) we obtain

$$|g_\omega(z)| < |e^{\omega z}| \cdot \varepsilon x^{-1} e^{-\lambda_n x} \cdot cx$$

$$= \varepsilon c e^{\omega x} e^{-\lambda_n x}$$

$$= \varepsilon c e^{(\omega - \lambda_n)x}$$

$$\leq \varepsilon c e^{Kx}$$

$$\leq \varepsilon c e^{K \frac{p}{2}} .$$

Therefore, by taking ω sufficiently large, and for any z such that z is on the perimeter of R and $Re(z) > 0$, we can make $|g_\omega(z)|$ as small as we like.

Similarly, for z on the perimeter of R and $Re(z) < 0$, we have:

(48) $$|h_\omega(z)| < \varepsilon(-x^{-1})e^{\lambda_n x}$$

Therefore, combining (46) and (48) we obtain

$$\begin{aligned}
|g_\omega(z)| &< |e^{\omega z}| \cdot \varepsilon(-x)^{-1} e^{\lambda_n x} \cdot c(-x) \\
&= \varepsilon c e^{\omega(-x)} e^{\lambda_n x} \\
&= \varepsilon c e^{(\omega-\lambda_n)(-x)} \\
&\leq \varepsilon c e^{K(-x)} \\
&\leq \varepsilon c e^{K\frac{p}{2}} .
\end{aligned}$$

Therefore, by taking ω sufficiently large, and for any z such that z is on the perimeter of R and $Re(z) < 0$, we can make $|g_\omega(z)|$ as small as we like. Since $g_\omega(s_1 i) = g_\omega(s_2 i) = 0$ and for the estimations above for the rest of the perimeter of R, then we have that

$$\lim_{\omega\to\infty} g_\omega(z) = 0$$

uniformly on the perimeter of R. Since g_ω is analytic on the rectangle R and its interior, then by the maximum modulus principle we have that

$$\lim_{\omega\to\infty} g_\omega(z) = 0$$

uniformly for z on the rectangle R or its interior, and therefore on the interval $[t_1 i, t_2 i]$. By the definition of g_ω, this means that the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$$

converges uniformly on the interval $[t_1 i, t_2 i]$. $\square$

BIBLIOGRAPHY

1. Abel, N. H., "Untersuchungen über die Reihe $1 + \frac{m}{1} x + \frac{m(m-1)}{1 \cdot 2} x^2 + \ldots$," *Journal für Reine und Angewandte Mathematik*, vol. 1, 1826, pp. 311-339.

2. Beke, Em., "Vizsgálalok az Analytikai Függvények Elmélete," *Math. es Termeszettud. Ertesitö*, vol. 34, 1916, pp. 1-61.

3. Bernstein, Vladimir, *Lecons sur les Progrés Récents de la Théorie des Séries de Dirichlet*, Imprimerie Gauthier-Villars, Paris, 1933.

4. Boas, Ralph P., Jr., *Entire Functions*, Academic Press Inc., Publishers, New York, N. Y., 1954.

5. Bohr, Niels,H., "Sur la Convergence des Séries de Dirichlet," *Comptes Rendus de l'Academie des Sciences de Paris*, vol. 151, 1910, pp. 375-377.

6. Bronšteǐn, B. S., "On Taylor-Dirichlet Series, *Soviet Math. Doklady*, vol. 6, no. 195, 1965, pp. 1385-1386.

7. Cahen, E., "Sur la Fonction $\zeta(s)$ de Riemann et sur les Fonctions Analogues," *Annales Science l'Ecole Normale*, series 3, vol. 11, 1894, pp. 75-164.

8. Carlson, Fritz and Edmund Landau, "Neuer Beweis und Verallgemeinerungen des Fabryschen Lückensatzes," *Göttingen Nachrichten*, 1921, pp. 184-188.

9. Cauchy, A. L., *Analyse Algébrique*, Paris, 1821.

10. Cauchy, A. L., "Mémoire sur la Mécanique Céleste et sur un Noveau Calcul Appelé Calcul des Limites," *Bulletin de Férussac*, vol. 15, 1831, pp. 260-269.

11. Chandrasekharan, K. and S. Minakshisundaram, *Typical Means*, Oxford University Press, London, 1952.

12. Cramer, Harald, "Un Théoreme sur les Séries de Dirichlet et son Application," *Arkiv för Matematik, Astronomi och Fysik*, vol. 13, no. 22, 1918, pp. 1-14.

13. Dienes, P., *The Taylor Series,* Oxford at the Clarendon Press, London, 1931.

14. Dirichlet, Peter G. L., "Recherches sur Diverses Applications de l'Analyse Infinitesimale à la Théorie des Nombres," *Journal für Reine und Angewandte Mathematik,* vol. 19, 1839, pp. 344-329.

15. Fabry, E., "Sur les Points Singuliers d'une Fonction Donnée par son Développement en Série et l'impossibilite du Prolongement Analytique dans des cas Très Généraux," *Annales Science l'Ecole Normale,* series 3, vol. 13, 1896, pp. 367-399.

16. Fatou, P., "Série Trigonométrique et Série de Taylor," *Acta Mathematica,* vol. 30, 1906, pp. 335-400.

17. Hardy, G. H. and Marcel Riesz, "The General Theory of Dirichlet's Series," *Cambridge Tracts in Mathematics and Mathematical Physics,* no. 18, 1952.

18. Jensen, J. L. W. V., "On Raekkers Konvergens," *Tidsskrift for Matematikk,* series 5, vol. 2, 1894, pp. 63-72.

19. Kojima, Tetsukô, "On the Convergence-Abscissa of General Dirichlet's Series," *Tohoku Mathematical Journal,* vol. 6, 1914-15, pp. 134-139.

20. Kuniyeda, Motoji, "Uniform Convergence—Abscissa of General Dirichlet's Series," *Tohohu Mathematical Journal,* vol. 9, 1916, pp. 7-27.

21. Kuniyeda, Motoji, "On the Abscissa of Summability of General Dirichlet's Series," *Tohoku Mathematical Journal,* vol. 9, 1916, pp. 245-262.

22. Landau, Edmund, "Über einen Satz von Tschebyschef," *Mathematische Annalen,* vol. 61, 1905.

23. Landau, Edmund, *Darstellung und Begründung einiger Neuerer Ergebnisse der Funktionentheorie,* Chelsea Publishing Company, New York, N. Y., 1929.

24. Landau, Edmund, "Über die Ronvergenz einiger Klassen von unendlichen Rechen am Rande des Konvergenzgebietes," *Monatsshefte für Math.,* vol. 18, 1907, pp. 8-28.

25. Mandelbrojt, S., "Dirichlet Series," *Rice Institute Pamphlet,* vol. 31, no. 4, 1938.

26. Markuschevich, A. I., *Theory of Functions of a Complex Variable,* vol. I, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965.

27. Moretti, Gino, *Functions of a Complex Variable,* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.

28. Prasad, B. N., "Presidential Address," to the Indian Mathematical Society, *Mathematics Student,* vol. 31, 1963, pp. 145-166.

29. Pringheim, A., "Über das Verhalten gewisser Potenzreihen auf dem Konvergenkreis," *Mathematische Annalen,* vol. 25, 1885, pp. 419-427.

30. Riesz, Marcel, "Sur la Sommation des Series de Dirichlet," *Comptes Rendus de l'Academie des Sciences de Paris,* vol. 149, 5 July, 1909, pp. 18-21.

31. Riesz, Marcel, "Neuer Beweis des Fatouschen Satzes," *Göttingen Nachrichten,* 1916, pp. 62-65.

32. Ritt, J. F., "On a General Class of Linear Homogeneous Differential Equations of Infinite Order with Constant Coefficients," *Transactions of the American Mathematical Society,* vol. 18, 1917, pp. 27-49.

33. Sansone, Giovanni and Johan Gerretsen, *Lectures on the Theory of Functions of a Complex Variable,* P. Noordhoff-Groningen, Netherlands, 1960.

34. Schnee, W., "Über Dirichletschen Reihen," *Rendiconti di Palermo,* vol. 27, 1909, pp. 87-116.

35. Szász, Otto, "Über Singularitäten von Potenzreihen und Dirichletshen Reihen am Rande des Konvergenzbereiches," *Mathematische Annalen,* vol. 85, 1922, pp. 99-110.

36. Tanaka, Chuji, "Note on Dirichlet Series (1), On the Singularities of Dirichlet Series (1)," *Tohoku Mathematical Journal,* series 2, vol. 3, 1951, pp. 285-291.

37. Tauber, A., "Ein Satz aus der Theorie der Unendlichen Reihen," *Monatshefte für Mathematik und Physik,* vol. 8, 1897, pp. 273-277.

38. Taylor, Brook, *Methodus Incrementorum Directa et Inversa,* London, 1715.